Which side up? Falling bread revisited.

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Abstract

The problem of falling buttered bread, though solved by several authors already, still remains inspiring and interesting. Not only because the results found in literature sometimes differ a bit, but the problem may well attract students at both high school and introductory university level. (Their motivation may be quite pragmatic: how to save the carpet in case a buttered toast starts to fall buttered side down from the table?) And it is worth discussing how these students may solve the problem. Nowadays, of course, using digital camcorder and freeware computer programs, they may easily study it experimentally.

The paper presents both a theoretical analysis of the problem (an analytical derivation of equations of motion and their numerical integration – i.e. falling bread simulation) and its experimental study using video recording and videoanalysis. The possibility of how to tackle the problem at the high school level is also discussed.

Comparison of our results and results of previous studies is presented; moreover, we explored also the influence of some further factors (moment of inertia of the toast and its initial velocity).

1. Introduction

In this article we investigate the problem of a slice of bread put at the edge of a table so that it falls down (to, presumably, precious carpet). Is it really going to fall buttered side down? And is this due to physics or due to some mysterious Murphy's Law?

Fig. 1: Falling Landau-Lifshits (Note: The book is not buttered…)

Our intention to calculate the motion of falling bread is rather new. It was motivated by a presentation of this problem (and its qualitative solution) at a TV show. But, of course, the problem was investigated in several articles already.

Matthews [1] was probably the first who tried to tackle it. He considered the initial phase of the motion (before the free fall) just as the rotation of a thin board (lamina) around the edge of the table. He argued that the board (i.e. the bread) looses contact with the table nearly immediately after it starts to slide.

Bacon et al. [2], after mentioning approaches and results of [1] on few other articles, made a more detailed calculation of the motion of (both thin and thick) bread. He has shown that to describe the problem correctly, the phase of sliding must be taken into account, and arrived to results that agree with experiments. He also used analysis of videos of falling bread to determine the angular velocity of the bread.

What did we do:

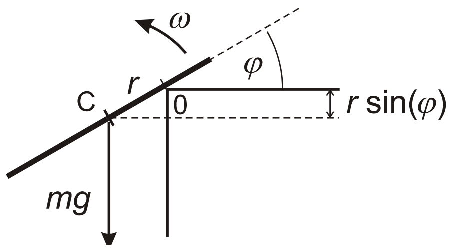
In this article we show how the problem of falling bread may be approached in a way suitable to be used as project for students starting even at a high school level. We took a slightly larger wooden board than authors mentioned above (of length l = 20 cm – in fact, a normal slice of Czech bread really has such dimensions) and put it at the edge of a table with a quite small overhang. (The bread just-just turned over the edge.) Then we investigated how the motion and landing result is influenced by various changes of parameters of the system, namely:

- Friction between "bread" – our wooden board – and the table. (To increase the friction coefficient we used sandpaper.)
- Moment of inertia J of the board – actually, its gyration radius J/m. (We used iron weights either near the centre or at the edges of the board to change J.)
- The initial horizontal velocity of the board. (In this case the fall is caused not by an overhang but by pushing the bread from the table with a considerable velocity.)

Our theoretical approach includes simple and rough estimate at high school level followed by numerical simulation of the motion of both thin and thick board. The simulation models all three phases of motion – rotation, sliding and free fall – and does not require any special modelling system. We also derive approximate analytical solution for the motion of a board pushed with a non-zero initial velocity. Finally, some simple considerations concerning scaling of the problem provide another view at some aspects of the motion.

2. Rough estimates at a high school level

Fig. 2



The computation of a free fall of the board after it left the table is a simple task even for high school students. The centre of mass moves down by a uniformly accelerated motion $y = \frac{1}{2} g t^2 + y_0$ and the board rotates with a constant angular velocity ω, so the angle φ is $\varphi = \omega t + \varphi_0$ – until the board hits the floor. It is sufficient to compute the angle φ at the time t_1 when the centre of mass C of the board is at the height a above the floor, a being the half-length of the board. (Below this height the board cannot change its position from "buttered side" up to this side down or vice versa.) If h is the height of the table, $(t_1)^2 = 2\frac{(h - y_0 - a)}{g}$ and, consequently, $\varphi_1 = \omega\sqrt{2\frac{(h - y_0 - a)}{g}} + \varphi_0$. If φ_1 is in the range 90°÷270°, i.e. $\frac{\pi}{2} < \varphi_1 < \frac{3\pi}{2}$, the board lands "buttered side down".

So it is clear that we need to estimate the angular velocity ω of the board after it left the table (and the angle φ_0 and the coordinate y_0 of the centre of mass at that instant). We can do this using the conservation of energy. If the centre of mass drops down by $r \sin\varphi$ (see Fig.2), the potential energy of the board drops by $mgr\sin\varphi$. This energy is converted to kinetic energy of the board – mainly to its rotational energy $\frac{1}{2} J\omega^2$. (The moment of inertia J should be taken with respect to the axis 0, but for a rough estimate we may use the moment J_C with respect to the centre of mass, since $J = J_C + mr^2$ and if r is small enough to the half-length a of the board $mr^2 << J_C = \frac{1}{3}ma^2$.) From these formulas we have

$$\omega = \sqrt{\frac{2mgr}{J}\sin\varphi} \doteq \sqrt{6\frac{g}{a}\frac{r}{a}\sin\varphi}\,. \qquad (1)$$

(Of course, if the board slides then some part of its mechanical energy is lost due to friction and its kinetic energy has also a part corresponding to the motion parallel to the board but this we neglect in our estimate.) We can expect (roughly) that the angle φ_0 at the instant of leaving the table is more than about 45° and less than about 70° (later we shall see this is a reasonable estimate.) so sin φ_0 should be between 0.7 and 0.95. For our board a = 0.1 m and $g \doteq 10\,\mathrm{m/s^2}$. The distance r of C and 0 may be about 1 cm (maybe after some sliding), so $r/a \doteq 0.1$. From the formula (1) we can then compute that, roughly, $\omega \doteq 7\,\mathrm{s^{-1}}$. The time of the free fall is about 0.36 s and we straightforwardly arrive at the conclusion that our board lands inevitably “buttered side down”.

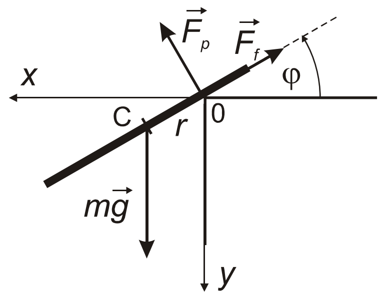
3. Beyond simple estimates: equations of motion of thin bread

The “full” equations of motion of a bread slice may be derived from the change of momentum and angular momentum of the bread:

$$m\ddot{\vec{r}}_C = m\vec{g} + \vec{F}_p + \vec{F}_f\,, \quad \dot{\vec{L}} = \vec{M}\,. \qquad (2)$$

$\vec{F}_p$ and $\vec{F}_f$ are forces by which the table edge acts to the board. $\vec{F}_p$ is perpendicular to the board, force $\vec{F}_f$,caused by friction, is parallel to it – se Fig. 3. For angular momentum only the component L_z perpendicular to x-y plane is relevant.

Fig. 3



Here we will consider just the case of a board with negligible thickness. And, of course, we will now describe only the part of motion when the board is in contact with the table (the following free fall is same as in previous section).

Putting $x_C = r\cos\varphi$, $y_C = r\sin\varphi$, $L_z = J_C\,\dot{\varphi}$, $M_z = r\,F_p$ (see Fig. 3) into equations (2) we arrive, after some rearrangements, to final equations

$$F_p = \frac{J_C}{J_C + mr^2}\, m\left(g\cos\varphi - 2\dot{r}\dot{\varphi}\right) \qquad (3a)$$

$$\ddot{\varphi} = \frac{r}{J_C}F_p \qquad (3b)$$

$$\ddot{r} = -\frac{F_f}{m} + r\,\dot{\varphi}^2 + g\sin\varphi \qquad (3c)$$

Note: Alternatively we could derive eqs. (3) from Lagrange equations. This derivation, being quite illustrative, will be published elsewhere. We could also take angular momentum and momenta of forces not to the centre of mass C but to the edge 0. (Students should realize that all these ways lead to the same equations!)

There are two phases of the motion before the board leaves the table:

a) rotation around the edge

In this case r = const. Equation (3c) then enables, after putting $\ddot{r} = 0$ into it, to determine the friction force F_f. This phase lasts while $F_f < f_s F_p$ where f_s is a coefficient of static friction. Then the board starts to slide.

b) sliding

During this phase the friction force is $F_f = f F_p$ (we consider the friction to be Coulomb). The force F_p given by (3a) decreases with time. When $F_p = 0$, the board ceases the touch with the edge and starts to fall and rotate freely.

The equations of motion of a thick board can be derived in a similar way. They are slightly more complicated – for the brevity we will not present them here but we use them in simulations described below.

4. Numerical simulation

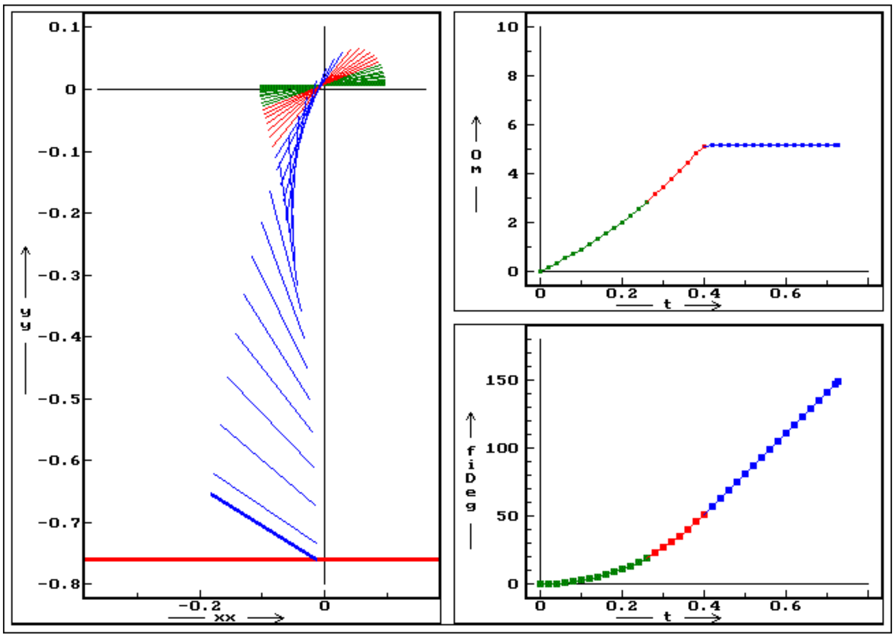
It is natural to solve eqs. (3) and similar equations for a thick board numerically. We tried to make the numerical simulation as simple and clear as possible and not to use any special system for solving differential equations. That's why we used the simple Euler method for solving (3). A part of the algorithm (for the sliding phase) illustrating the simulation is shown in the box. Here, `Om` and `afi` stands for $\dot{\varphi}$ and $\ddot{\varphi}$, `vr` and `ar` for $\dot{r}$ and $\ddot{r}$. Some auxiliary commands are omitted for brevity. The simulation algorithm was in fact written and run in an old DOS-based computational system *Famulus* but it may be easily rewritten into any programming language like Pascal, Basic etc. or into a modelling system.

```
REPEAT
  Fp = m*JC/(JC+m*r*r)*(g*cos(fi)-2*vr*Om)
  Ff = f*Fp
  afi= (r/JC)*Fp
  ar = -Ff/m + r*Om*Om + g*sin(fi)
  Om = Om + afi*dt
  fi = fi + Om*dt
  vr = vr + ar*dt
  r  = r + vr*dt
  t  = t + dt
  xC = r*cos(fi); yC = r*sin(fi);  DISP
UNTIL Fp<=0
```

The used time step `dt` was typically 0.2 ms but 1 ms proved to be quite sufficient. The precision of the numerical simulation was checked by monitoring total energy of the board. It is constant during the rotation phase and decreases in the sliding phase – but its decrease can be compared with the work done by the force F_f. The computed position of the board was displayed every 20 ms to enable easy comparison with experimental data.

The initial conditions used in the majority of simulations were simple: $\varphi(0)=0$, $\omega(0)=0$, $r(0)=r_0$ (small initial overhang, typically several mm), $\dot{r} = 0$. During the simulation values of various quantities can be plotted. The final angle φ_{landing} and the angular velocity ω during the free fall were always taken as the main output. The ω values were also used to compare our simulations with the results of Bacon et. al. [2]. For the dimensions of the board used by these authors our simulation for both thin and thick board produced the results in a complete agreement with those presented graphically at Fig. 7 in [2].

Fig. 4: An example of graphical outputs from our simulations



On Fig. 4, both positions of the board and graphs $\omega(t)$ (top one) and $\varphi(t)$ (bottom one) are plotted. The positions and values plotted in green correspond to the rotational phase of the motion, those plotted in red to the sliding phase and those in blue to the free fall.

5. Experiments and measurements

The processes of sliding and falling of the bread slice are too fast to be observed quantitatively without some form of data logging. In our case the datalogger was a standard digital camcorder. We followed a process that is sometimes called “the videoanalysis” – we took numerous clips containing fall of the board and then step by step measured positions of the board on each frame of the clip.

Fig.5: Different positions of the board (images extracted from one clip used for measurement)

A standard camcorder takes 25 frames per second, so a delay between two subsequent frames should be 0.04 s. In fact, these frames are interlaced which means that the camcorder takes twice a set of only one half of all lines, called one field, (e.g. even lines at first and then, after 1/50 s, the odd lines). For purpose of this experiment we separated these pairs of fields into following frames and received a clip with half pixel resolution in vertical dimension, but with 50 frames per second.

On the modified clip we measured the positions of three significant points on the board during its entire movement – the centre of mass (centre of the board) and the two ends of the board. The data were measured using the *Viana* [3] software, however any other free software tool for videoanalysis could have been used.

6. Processing the data

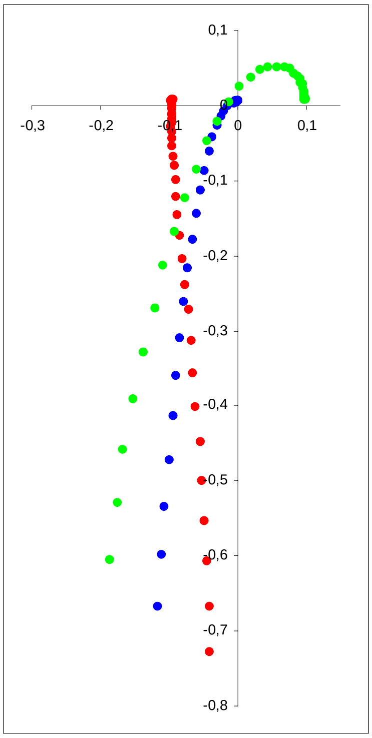
The measured data from *Viana* were exported into MS Excel and processed. For each position we calculated the distance between the centre of mass and the axis of rotation (i.e. the edge of the table), the distance between the board and the axis and angular velocity. From these values we could separate the three phases of the movement described in paragraph 3.

a) At the phase of rotation the distance of the centre of mass and the axis doesn't change significantly.

b) At the phase of sliding, the distance between the centre of mass and the axis increases but the board is still in contact with the axis. This can be recognized from the fact that the distance between the board (presented by a line containing the three measured points) and the axis is constant.

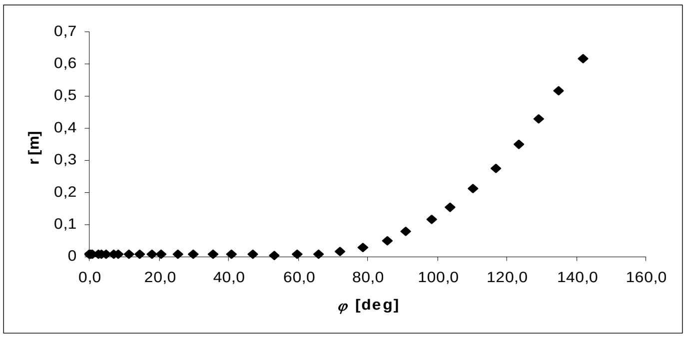
c) At the third phase, the free fall, the centre of mass moves with uniform acceleration g and the angular velocity of the board doesn't change.

Fig. 5: The set of data from one measurement



Each threesome of coloured points represents the position of the board at one instant. The axes are scaled in meters.

Fig. 6: Distance from axis vs. angle.



The board left the table at an angle of about 60°-65°.

7. Some results

Our experiments showed that both phases b) and c) occur later than the theory and numerical model predicts. The problem is that in the model and theory we can exactly find the beginnings of the phases, but during the experiments these changes are slow and it is hard to recognize that they occurred before the changes are significantly visible. The precise recognition of the phase starts is not impossible but it is necessary to be very careful and aware of this possible source of problems.

Table 1: Typical resulting values for different initial conditions

	ω [1/s]		$\varphi_{landing}$ [deg]	
	exp.	sim.	exp.	sim.
plain board on table	5,3	5,2	140	135
high friction	4,1	4,7	140	153
small J	7,2	7,5	189	195
big J	4,2	4,2	110	114

Besides these resulting parameters, we also compared the durations of phases and the angles between the table and the board at the moment of phase changes, and. The correspondence of the model and the experiment is quite satisfying. We did not expect precise correspondence because there are further facts that influence the motion, like the air resistance or radius of the table edge, that are not included in the model. Though, the character of the motion is described satisfyingly.

For the case of high friction, the difference between the model and experiment is quite big. We expect this to be an effect of energy losses while the grains of the sandpaper interact with the wooden board, however, precise understanding would require further study of the problem.

From Table 1 it can be seen that in our conditions the friction between the table and the board doesn't influence the result significantly. It influences the character of the motion because if the friction is high, the phase of rotation is very long (the angle at which the board leaves the table is quite big) and the following slipping phase is short. Still, due to the combination of big angle and lower angular velocity, the board falls butter-side down.

What does really matter is (expectedly) the moment of inertia J. A board with a small J (with two iron weights near the centre) gains during the fall a reasonably higher angular velocity than a plain board, the angular velocity of a board with big J (weights near the ends of the board) is lower.

Unluckily, if the board was a slice of bread, it would still land butter-side down regardless of the friction or the moment of inertia. To avoid this in some cases, we would have to use a higher (in case of small J) or smaller table (in case of big J).

8. Motion with non-zero initial velocity

If we push the board with some initial horizontal velocity, there is no rotational phase of motion. We ran the simulation adapted to this case to find for which initial velocity the board would land "buttered side" up. For f = 0.2 we found it was for velocities greater than 0.9 m/s. Greater f slightly increased the required velocity – for f = 0.45 the minimal velocity was 1.0 m/s.

We also compared the results of simulations with measurements for initial velocities of about 0.7 m/s, 0.95 m/s, 1.15 m/s and 1.5 m/s and found a good agreement. In the first case the "bread" really landed "buttered side" down, in other cases it was this side up. However, the final landing angles being between 55° and 85°, being it falling sandwiches, it still would not be very good for your carpet…)

It is interesting that in this case, especially for fast initial horizontal movement, an approximate formula for ω (and φ_{landing}) can be easily derived.

Let's expect the horizontal velocity υ_x to be constant. (The experiment shows that the decrease of υ_x due to friction is really small – for $\upsilon_x > 1$ m/s it decreases by less than 5%.) So, the x coordinate of the centre of mass increases uniformly with t: $x_C = \upsilon_x t$. The momentum of force that starts to rotate the board is therefore $M = mg\,x_C = mg\,\upsilon_x t$. For the change of angular momentum we have

$$\frac{\mathrm{d}(J\omega)}{\mathrm{d}t} = M = mg\,\upsilon_x t \ ,$$

so (because $\omega(0) = 0$):

$$J\omega = \int_0^t mg\,\upsilon_x t\,\mathrm{d}t = \tfrac{1}{2} mg\,\upsilon_x t^2 = \frac{mgx^2}{2g} \ . \qquad (4)$$

When the board leaves the table, it is approximately $x \doteq a$ (the half-length of the board – it is a bit less due to small rotation of the board, but we will neglect it). The moment of inertia at this point is $J = J_C + ma^2 = \tfrac{1}{3}ma^2 + ma^2 = \tfrac{4}{3}ma^2$. From this and (4), finally we get

$$\omega = \frac{3}{8}\frac{g}{v_x} \quad . \tag{5}$$

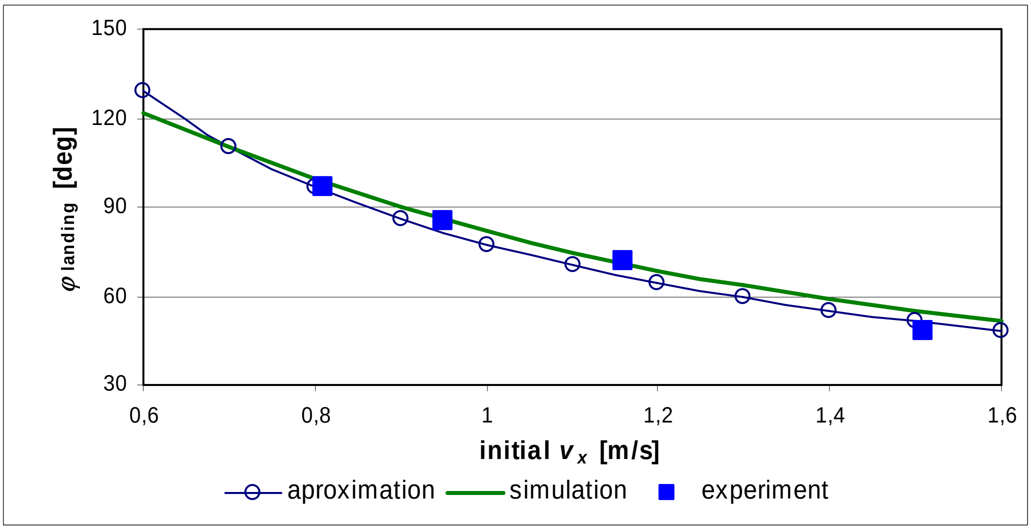
If we take for the time of the free fall the approximate value $T = \sqrt{2\frac{h-a}{g}}$ (ignoring the small initial velocity in y direction and taking h–a instead of h for reasons discussed above in part 2) and if we take the angle φ at the instant when the board leaves the table as approximately zero, we get for the final angle at which the board hits the floor

$$\varphi_{\text{landing}} \doteq \omega T \doteq \frac{3\sqrt{2}}{8}\sqrt{(h-a)g}\,\frac{1}{v_x} \quad . \tag{6}$$

If the board should not rotate for more than $\frac{\pi}{2}$ (to land yet "buttered side" up), we get from $\varphi_{\text{Landing}} < \frac{\pi}{2}$ and (6) the condition $v_x > \frac{3\sqrt{2(h-a)g}}{4\pi}$.

Fig. 6 shows how simple analytic formula (6) approximates results of numerical simulation and how it corresponds to the data from experiments.

Fig. 6 – comparison of approximation, model and results



9. Further considerations

a) How would our bread fall on the Moon?

The simulation also enables us to investigate how experiments would result in conditions we cannot easily establish – for example, for different gravitational acceleration g. It comes out from the model that angular velocity ω would differ but the final angle of landing φ_{landing} would be the same for any g. So, other parameters being the same, the bread on the Moon will fall on the ground with the same angle as on the Earth.

In fact, we can understand (or predict) this behaviour directly from equations (3). If we scale g by a constant factor k to $g_{\text{new}} = k \cdot g$ and time t to $t_{\text{new}} = t/\sqrt{k}$ (and all forces from F to $F_{\text{new}} = k \cdot F$) the equations (3) stay the same.

Due to the scaling of t, the angular velocity $\omega = \dot{\varphi}$ scales as $\omega_{\text{new}} = \sqrt{k} \cdot \omega$ (and $\ddot{\varphi}$ as $\ddot{\varphi}_{\text{new}} = k \cdot \ddot{\varphi}$ and similarly $\dot{r}$ and $\ddot{r}$), but φ stays the same, $\varphi_{\text{new}} = \varphi$. So, on the Moon where

$g_{\text{new}} \doteq g/6$ everything takes $\sqrt{6}$ -times longer but the final position and angle are the same as in our lab on the Earth.

b) Scaling the lengths

Similarly we may ask what will happen if we scale all the lengths of the problem, i.e. if we change *a, h* and *r* to $a_{\text{new}} = K \cdot a$, $h_{\text{new}} = K \cdot h$ and $r_{\text{new}} = K \cdot r$ where *K* is constant. It can be easily seen that equations (3) stay the same if we scale also the time *t* to $t_{\text{new}} = \sqrt{K} \cdot t$. (The angular velocity then scales as $\omega_{\text{new}} = \omega / \sqrt{K}$.)

So, by taking an about 2-times longer board than the authors of [2] did, we took the advantage of about 1.4-times lower ω what, together with 1.67-times higher frame rate, enabled easier and (at least potentially) more precise measurements of positions of the board at all phases of its motion. In fact, the above mentioned scaling argument suggests a possibility of further improvement of measurements in the initial phases of the motion. If we take a board of a length 80 cm, all time intervals corresponding to the same angles of rotation would be twice as long as in the case of our 20 cm board and, correspondingly, angular velocities would be halved. And, at the same time, the angles at which the board starts to slide and leaves the table should stay unchanged. This would ease taking more precise data from experiments.

Taking both two scalings together suggests yet another possibility of presenting the whole problem. It is clear that in a sufficiently large "mechanical model" of falling bread in a lower *g* (which could be mimicked by an above pointing vertical force acting in the centre of mass of the board) the initial phases of the motion could be clearly seen even without a video analysis, just by a naked eye.

10. Conclusions

We analyzed the problem of bread (a board) falling from a table and turning butter-side down analytically and numerically, studied experimentally the motion of a falling board and compared the results. The character of experimentally observed motion is the same as what the numerical models predict. On the other hand, the correspondence is not perfect due to further phenomena that are not included in the model.

When compared to previous studies, our approach is new in studying the effects of changing parameters of the motion (moment of inertia of the board, initial velocity, friction). Innovative is also the idea of recognizing and separating the three phases of the motion that was performed in both model and experiment and used for their comparison. Our experimental method is more precise and both experiment and model were made using common or free software tools.

There are further interesting problems to study – the experiment could be done with a larger board (to improve precision, see above) to examine the details of initial phases of the motion. Another improvement would be adding the shape of the table edge to the model

We think that the problem of falling bread slice might be extended into a quite interesting project or a lab for students, hopefully in two variants – at high school and university level.

List of references

[1] Matthews R. A. J.: Tumbling toast, Murphy's Law and the fundamental constants. *Eur. J. Phys.* 16 (1995), 172-176.

[2] Bacon M. E., Heald G., James M.: A closer look at tumbling toast. *Am. J. Phys.* 69 (Jan. 2001), 38-43.

[3] VIANA (automatische VIdeoANAlyse): http://didaktik.physik.uni-essen.de/viana/