1. Diferenciální vektorové operátory

Hamiltonův operátor, nabla

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \qquad \vec{i}.\vec{j} = \delta_{ij}, \qquad \vec{i}\times\vec{j} = \vec{k}, \qquad \vec{j}\times\vec{k} = \vec{i}, \qquad \vec{k}\times\vec{i} = \vec{j}$$

gradient

$$\vec{\nabla}\varphi = \text{grad }\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

převádí skalární pole na vektorové pole

divergence

$$\vec{\nabla}.\vec{A} = \text{div }\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

převádí vektorové pole na pole skalární

rotace

$$\vec{\nabla}\times\vec{A} = \text{rot }\vec{A} = \vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

převádí vektorové pole na jiné vektorové pole

laplace

$$\Delta = \vec{\nabla}.\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\Delta\vec{A} = \vec{i}.\Delta A_x + \vec{j}.\Delta A_y + \vec{k}.\Delta A_z$$

Několik identit

$$\text{div grad } s = \vec{\nabla}.\vec{\nabla}s = \Delta s \qquad \text{rot grad } s = \vec{\nabla}\times\left(\vec{\nabla}s\right) = 0$$

$$\text{div rot }\vec{A} = \vec{\nabla}.\left(\vec{\nabla}\times\vec{A}\right) = 0 \qquad \text{rot rot }\vec{A} = \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \text{grad div }\vec{A} - \Delta\vec{A}$$

Geometrická definice operátorů

$$\text{div }\vec{F} = \lim_{V_i\to 0}\frac{1}{V_i}\oint_{S_i}\vec{F}.\vec{n}da_i$$

tok vektoru $\vec{F}$ plochou uzavírající elementární objem V_i

$$\vec{n}.\text{rot }\vec{F} = \lim_{a_i\to 0}\frac{1}{a_i}\oint_{C_i}\vec{F}.d\vec{s}$$

cirkulace vektoru $\vec{F}$ křivkou omezující elementární plochu a_i

Gaussův – Ostrogradského teorém

$$\Phi_c = \oint_S \vec{F}.\vec{n}da = \int_V \text{div }\vec{F}dV$$

Stokesův teorém

$$\Gamma = \oint_c \vec{F}.d\vec{s} = \int_S \text{rot }\vec{F}.\vec{n}da$$

Křivočaré souřadnice

Válcové (cylindrické) souřadnice

$q_1 = \rho\left(0,\infty\right)$	$x = \rho\cos\varphi$	$\rho = \sqrt{x^2 + y^2}$
$q_2 = \varphi\left(0,2\pi\right)$	$y = \rho\sin\varphi$	$\varphi = \text{arctg}\frac{y}{x}$
$q_3 = z\left(-\infty,\infty\right)$	$z = z$	$z = z$

Kulové (sférické) souřadnice

$q_1 = r(0,\infty)$ $\quad$ $x = r \sin\vartheta \cos\varphi$ $\quad$ $r = \sqrt{x^2 + y^2 + z^2}$

$q_2 = \vartheta(0,\pi)$ $\quad$ $y = r \sin\vartheta \sin\varphi$ $\quad$ $\vartheta = \arccos \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

$q_3 = \varphi(0,2\pi)$ $\quad$ $z = r \cos\vartheta$ $\quad$ $\varphi = \operatorname{arctg} \dfrac{y}{x}$

Vektorové operátory v křivočarých souřadnicích

Válcové souřadnice

$$\text{grad } u = \vec{\nabla}.u = \frac{\partial u}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \varphi}\vec{e}_\varphi + \frac{\partial u}{\partial z}\vec{e}_z$$

$$\text{div } \vec{A} = \vec{\nabla}.\vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\text{rot } \vec{A} = \vec{\nabla}\times\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\vec{e}_\varphi + \left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\vec{e}_z$$

$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}$$

Kulové souřadnice

$$\text{grad } u = \vec{\nabla}u = \frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial \vartheta}\vec{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial u}{\partial \varphi}\vec{e}_\varphi$$

$$\text{div } \vec{A} = \vec{\nabla}\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta\, A_\vartheta)}{\partial \vartheta} + + \frac{1}{r\sin\vartheta}\frac{\partial A_\varphi}{\partial \varphi}$$

$$\text{rot } \vec{A} = \vec{\nabla}x\vec{A} = \frac{1}{r\sin\vartheta}\left(\frac{\partial(\sin\vartheta\, A_\varphi)}{\partial \varphi} - \frac{\partial A_\vartheta}{\partial \varphi}\right)\vec{e}_r + \left(\frac{1}{r\sin\vartheta}\frac{\partial A_r}{\partial \varphi} - \frac{1}{r}\frac{\partial(r\, A_\varphi)}{\partial r}\right)\vec{e}_\vartheta + \frac{1}{r}\left(\frac{\partial(r\, A_\vartheta)}{\partial r} - \frac{\partial A_r}{\partial \vartheta}\right)\vec{e}_\varphi$$

$$\Delta u = \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial u}{\partial \vartheta}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial^2 u}{\partial \varphi^2}$$

Několik příkladů použití diferenciálních operátorů

$\vec{\nabla}(u.v) = u.\vec{\nabla}v + v\vec{\nabla}u$ $\quad$ $\vec{\nabla}(u.\vec{A}) = u.\vec{\nabla}.\vec{A} + \vec{A}.\vec{\nabla}u$

$(\vec{A}.\vec{\nabla})\vec{r} = \vec{A}$ $\quad$ $(\vec{r} = \vec{e}_x.x + \vec{e}_y.y + \vec{e}_z.z)$

$(\vec{\nabla}.\vec{A})\vec{B} = (\vec{A}.\vec{\nabla}).\vec{B} + \vec{B}(\vec{\nabla}.\vec{A})$

$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{C}\times\vec{B})\times\vec{A} = \vec{B}.(\vec{A}.\vec{C}) - \vec{C}.(\vec{A}.\vec{B})$

$\vec{\nabla}.(\vec{A}\times\vec{B}) = \vec{B}.(\vec{\nabla}\times\vec{A}) - \vec{A}.(\vec{\nabla}\times\vec{B})$

$\vec{\nabla}\times(u.\vec{A}) = u.(\vec{\nabla}\times\vec{A}) - \vec{A}\times\vec{\nabla}u$

$\vec{\nabla}.(\vec{A}.\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}.\vec{\nabla})\vec{B} + (\vec{B}.\vec{\nabla})\vec{A}$

$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B} + \vec{A}.(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{A})$